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Perturbative dynamics of matrix string for the membrane

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Abstract: Recently Sekino and Yoneya proposed a way to regularize the world volume theory of membranes wrapped around S^1 by matrices and showed that one obtains matrix string theory as a regularization of such a theory. We show that this correspondence between matrix string theory and wrapped membranes can be obtained by using the usual M(atrix) theory techniques. Using this correspondence, we construct the super-Poincaré generators of matrix string theory at the leading order in the perturbation theory. It is shown that these generators satisfy 10 dimensional super-Poincaré algebra without any anomaly.

Keywords: M-Theory, M(atrix) Theories, Space-Time Symmetries, Anomalies in Field and String Theories.

Contents

1. Introduction

In order to define the world volume theory of membranes, one needs a way to regularize the ultraviolet divergences. The matrix regularization is the most useful way to do so after the light-cone gauge fixing of the world volume diffeomorphism [1, 2] . The dependence on the two spatial world volume coordinates are represented by matrices, and the regularized theory becomes a quantum mechanical system described by a set of matrix variables. The area preserving diffeomorphism, which is the residual symmetry in the light-cone gauge, is replaced by U(N) gauge symmetry and the resulting theory becomes a one-dimensional gauge system. Later it was shown that the same system can be obtained from string theory, trying to describe the DLCQ (discretized light-cone quantization) limit of M theory [3, 4, 5].

The above matrix quantum mechanics provides a regularization of *unwrapped* membranes. In M theory [6] , one is tempted to consider *wrapped* membranes, i.e., membranes with one spatial world volume direction wrapped around a compactified direction in the space-time, which become fundamental strings when the compactification radius is taken to be very small. Wrapped membranes were first studied in detail in ref. [7]. Recently, Sekino and Yoneya gave a recipe [8] to regularize the world volume theory of wrapped membranes by using matrices. They showed that by applying their rules one obtains matrix string theory [9] as a regularization of the world volume theory of wrapped membranes. (See ref. [10, 11, 12] for related works, and ref. [13] for recent other approaches to the membrane theory.).

Matrix string theory was originally obtained from M(atrix) theory [3] via the procedure in ref. [14], in order to describe M theory with one space direction compactified. (See the appendix of ref. [15] for details.). Since it was derived following the logic of string theory and M theory, it is likely that the matrix regularization of Sekino and Yoneya can also be rederived from string theory and M theory as in the unwrapped case. In the first half of this paper, we would like to show that it is indeed the case and the matrix regularization rule of Sekino and Yoneya can be rederived using the familiar techniques in M(atrix) theory.

Therefore, matrix string theory (or a two dimensional super Yang-Mills theory) can be considered as a microscopic description of the world volume theory of wrapped membranes in M theory. In the Sekino-Yoneya regularization, each matrix element of the matrix variables corresponds to a tiny string bit on the world volume of the membrane. Therefore in order to get information on the world volume theory of the wrapped membranes, we need to analyze the infrared properties of the two dimensional U(N) super Yang-Mills theory. Here, we would like to concentrate on the ultraviolet region where we can calculate various quantities perturbatively. The quantities which we are particularly interested in are the anomalies in the super-Poincaré algebra. Requiring that such anomalies vanish, we can determine the critical dimension of matrix string theory.

In the latter half of this paper, we will construct the super-Poincaré generators of matrix string theory using its relation with wrapped membranes. We see that at the leading order of the perturbative expansion the theory can be described as a theory of N strings without any interactions between them. At this order, the membrane appears to be resolved into the multi-strings put at the sites along the compact '11th' direction [16]. As a consequence, the super-Poincaré generators are the sum of those for these strings and they satisfy the super-Poincaré algebra when the number of space-time dimensions is equal to 10 for the strings. Since the membrane is wrapped around one direction, the membrane lies in 11 dimensional space-time.

The organization of this paper is as follows. In section 2, we briefly review the matrix regularization of wrapped membrane action proposed by Sekino and Yoneya. In section 3, we show that the rule given by Sekino and Yoneya can be derived using the usual M(atrix) theory techniques. In section 4, we will examine the super-Poincaré invariance of matrix string theory using its relation to the wrapped membrane. We will show that the super-Poincaré generators of the theory can be written mostly as the sum of those of N independent type-IIA strings and that there exists super-Poincaré symmetry at the lowest order in perturbation theory. Section 5 is devoted to conclusions and discussions.

2. Matrix regularization of wrapped membranes

We recall briefly the direct construction of the map of the membrane variables to the matrices in matrix string theory á la Sekino and Yoneya [8] after fixing the notation used throughout this paper.

2.1 light-cone gauge fixing

Here we recall the conventional light-cone gauge action for the supermembrane. The Nambu-Goto action for the supermembrane propagating in 11-dimensional Minkowski space-time is[1]

$$I_P = -\int d^3\sigma \sqrt{-g} + I_{\rm WZ}\,, \tag{2.1}$$

with the Wess-Zumino term $I_{\rm WZ}$ given by

$$I_{\rm WZ} = \int d^3\sigma\, \frac{i}{2}\epsilon^{ijk}(\bar\theta\Gamma_{\mu\nu}\partial_i\theta)\left(-\Pi_j^{\ \mu}\Pi_k^{\ \nu} - i\Pi_j^{\ \mu}(\bar\theta\Gamma^\nu\partial_k\theta) + \frac{1}{3}(\bar\theta\Gamma^\mu\partial_j\theta)(\bar\theta\Gamma^\nu\partial_k\theta)\right). \tag{2.2}$$

Here g_{ij} is

$$g_{ij} = \Pi_i^{\ \mu}\Pi_j^{\ \nu}\eta_{\mu\nu}\,, \tag{2.3}$$

with the invariant 1-form $\Pi_i^{\ \mu}$ defined by

$$\Pi_j^{\ \mu} = \partial_j X^\mu - i\bar\theta\Gamma^\mu\partial_j\theta\,. \tag{2.4}$$

θ is a Majorana fermion and $\bar\theta = \theta^T C$ where C is the charge conjugation matrix in 11 dimension. The membrane tension is set equal to 1 here.

In this paper, we would like to study the membrane with its one spatial direction, say, $\sigma^2 \approx \sigma^2 + 2\pi$, wrapped once along the eleventh coordinate $x^\# \approx x^\# + 2\pi R$. The membrane coordinate $X^\#$ satisfies

$$X^\#(\sigma^0,\, \sigma^1,\, \sigma^2+2\pi) = X^\#(\sigma^0,\, \sigma^1,\, \sigma^2) + 2\pi R\,. \tag{2.5}$$

Such a coordinate can be rewritten as

$$X^\#(\sigma) = R\left(\sigma^2 - A_1(\sigma)\right), \tag{2.6}$$

by using A_1 obeying the periodic boundary condition

$$A_1(\sigma^0,\, \sigma^1,\, \sigma^2+2\pi) = A_1(\sigma^0,\, \sigma^1,\, \sigma^2)\,. \tag{2.7}$$

From the action, one can obtain the conjugate momenta P_μ, P_{A_1} and χ_α of X^μ ($\mu = 0, \cdots, 9$), A_1 and θ^α respectively. It is also easy to identify the hamiltonian constraint $\phi_0 \approx 0$ and the momentum constraints $\phi_r \approx 0$ ($r = 1,\, 2$) as well as the fermionic constraints $\xi_\alpha \approx 0$ corresponding to the kappa symmetry.

As usual, we decompose X^μ, for instance, into the light-cone directions

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^9 \pm X^0\right), \tag{2.8}$$

and the transverse directions X^I ($I = 1, \ldots, 8$). Now we employ the light-cone gauge fixing

$$\begin{aligned} P^+ &= \text{constant}\,, \\ X^+ &= \sigma^0\,, \\ \Gamma^+\theta &= 0\,. \end{aligned} \tag{2.9}$$

[1] We follow the convention in ref. [17].

By using the representation for gamma matrices

$$\begin{aligned} \Gamma^0 &= i\sigma_2 \otimes \mathbf{1}_{16} = C\,, \qquad & \Gamma^{\#} &= \sigma_3 \otimes \gamma^9\,, \\ \Gamma^I &= \sigma_3 \otimes \gamma^I\,, & \Gamma^9 &= \sigma_1 \otimes \mathbf{1}_{16}\,, \end{aligned} \tag{2.10}$$

where γ^j $(j = 1, \ldots, 9)$ are the gamma matrices in 9-dimensional euclidean space, together with an appropriate rescaling, we get the action which looks like that of a two dimensional gauge theory as

$$\begin{aligned} I_0^{\rm lc} = \int dt d^2\vec{\sigma} \Big[& \frac{R^3}{2}(F_{t1})^2 + \frac{1}{2}(D_t X^I)^2 - \frac{1}{2}(D_1 X^I)^2 - \frac{1}{4}\frac{1}{R^3}\left(\{X^I,\, X^J\}\right)^2 - \\ & - \frac{i}{2}\,\Psi^T D_t \Psi - \frac{i}{2}\,\Psi^T \gamma^9 D_1 \Psi + i\frac{1}{R^{3/2}}\,\frac{1}{2}\,\Psi^T \gamma^I \{X^I,\, \Psi\}\Big]\,. \end{aligned} \tag{2.11}$$

Here the time-like coordinate t on the world volume is defined as

$$t \equiv R\sigma^0\,, \tag{2.12}$$

and Ψ denotes the fermionic variables left after the light-cone gauge fixing;

$$\theta = \frac{1}{2^{\frac{3}{4}}\sqrt{P^+}}\begin{pmatrix}\Psi \\ 0\end{pmatrix}\,. \tag{2.13}$$

The bracket $\{,\}$ denotes

$$\{\Phi_1,\, \Phi_2\} = \epsilon^{rs}\partial_r \Phi_1 \partial_s \Phi_2\,, \tag{2.14}$$

with $\epsilon^{rs} = -\epsilon^{sr}$, $\epsilon^{12} = 1$, while the covariant derivative and the field strength are defined by

$$\begin{aligned} D_a \Phi &\equiv \partial_a \Phi + \{A_a,\, \Phi\} \qquad (a = 0,\, 1)\,, \\ F_{01} &\equiv \partial_0 A_1 - \partial_1 A_0 + \{A_0,\, A_1\}\,. \end{aligned} \tag{2.15}$$

One important remark here is that the kinetic terms of X^I do not involve the σ^2 derivatives. In the case of the unwrapped membrane, the kinetic terms do not include the σ^1 and σ^2 derivatives. We need some regularization to make sense out of such a theory. The matrix regularization replaces the dependence on the spatial coordinates by matrix degrees of freedom, and the regularized theory becomes a matrix quantum mechanics. In our case, we expect that some two dimensional theory should regularize the wrapped membrane.

2.2 Sekino-Yoneya procedure

Sekino and Yoneya found a way to map the variables of the wrapped membrane action (2.11) to the variables in matrix string theory. Let us define $\sigma \equiv \sigma^1$ and $\rho \equiv \sigma^2$. Now we will discretize the coordinate ρ, which corresponds to the wrapped direction, so that it takes only the following values: $\rho \in \{0, \frac{2\pi}{N}, \ldots, \frac{2\pi}{N}(N-1)\}$. For each variable $\Phi(\sigma, \rho)$ appearing in the membrane action, we Fourier decompose it as

$$\Phi(\sigma,\, \rho) = \sum_n e^{in\rho}\, \Phi_n(\sigma)\,. \tag{2.16}$$

Since ρ is discretized, n is restricted to $|n| \le N/2$. From such Fourier modes, one can define an $N \times N$ matrix variable $\phi_{kl}(\theta)$ $(k, l = 1, \ldots, N)$ following the boundary condition

$$\phi_{kl}(\theta + 2\pi) = \phi_{k+1,\,l+1}(\theta)\,. \tag{2.17}$$

The correspondence between $\Phi_n(\sigma)$ and $\phi_{kl}(\theta)$ is given as [8]

$$\Phi_{k-l}(\sigma_{kl}(\theta)) = \phi_{kl}(\theta)\,, \tag{2.18}$$

where $\sigma_{kl}(\theta)$ is given by[2]

$$\sigma_{kl}(\theta) \equiv \frac{(k-1)+(l-1)}{N}\,\pi + \frac{\theta}{N}\,. \tag{2.19}$$

Therefore, in the matrix description, the σ-direction on the world volume of the membrane is divided into N segments and each matrix element corresponds to each segment. The Kaluza-Klein modes $(n \neq 0)$ along the wrapped direction ρ are packed into the off-diagonal elements of the matrix variables $(k \neq l)$.

To see that the light-cone membrane action (2.11) is indeed mapped to the matrix string action, it is necessary to rewrite the bracket $\{,\}$ and the spatial integral $\frac{2\pi}{N}\sum_\rho \int d\sigma$ in terms of matrices. It is straightforward to check the correspondence

$$\begin{aligned}
\frac{1}{N}\sum_\rho e^{-i(k-l)\rho}\{\Phi_1,\, \Phi_2\} &\mapsto i\left(\frac{N}{2\pi}\right)[\phi_1,\, \phi_2]_{kl}\,,\\
\frac{1}{N}\sum_\rho \int_0^{2\pi}\frac{d\sigma}{2\pi} &\mapsto \int_0^{2\pi}\frac{d\theta}{2\pi}\,\frac{1}{N}\,\mathrm{tr}\,,\\
\partial_\sigma &\mapsto N\partial_\theta\,,\\
A_\sigma &\mapsto a_\theta\,,
\end{aligned} \tag{2.20}$$

which is valid in the large-N limit. By using those relations, the action (2.11) is replaced by

$$\begin{aligned}
I_{\mathrm{WM}} = \int dt \int_0^{2\pi}\frac{d\theta}{2\pi}\frac{1}{N}\,\mathrm{tr}\Bigg[&\frac{R^3}{2}(f_{t\,\theta})^2 + \frac{1}{2}(D_t x^I)^2 - \frac{N^2}{2}(D_\theta x^I)^2 - \frac{N^2}{(2\pi)^2 R^3}\frac{1}{4}\left(i[x^I,\, x^J]\right)^2 -\\
&-\frac{i}{2}\,\psi^T\rho^0 D_t\psi - N\frac{i}{2}\,\psi^T\gamma^9 D_\theta\psi - \frac{N}{2\pi R^{3/2}}\,\frac{1}{2}\,\psi^T\gamma^I[x^I,\, \psi]\Bigg]\,,
\end{aligned} \tag{2.21}$$

with

$$f_{t\,\theta} \equiv \partial_t a_\theta - N\partial_\theta a_t + i\frac{N}{2\pi}\,[a_t,\, a_\theta]\,. \tag{2.22}$$

By using the rescaled time-like coordinate τ defined by

$$\tau \equiv N\,t\,, \tag{2.23}$$

the above action is rewritten as

$$\begin{aligned}
I_{\mathrm{MS}} = \int d\tau \int_0^{2\pi}\frac{d\theta}{2\pi}\,\mathrm{tr}\Bigg[&\frac{R^3}{2}(f_{\tau\,\theta})^2 + \frac{1}{2}(D_\tau x^I)^2 - \frac{1}{2}(D_\theta x^I)^2 - \frac{1}{(2\pi)^2 R^3}\frac{1}{4}\left(i[x^I,\, x^J]\right)^2 -\\
&-\frac{i}{2}\,\psi^T D_\tau\psi - \frac{i}{2}\,\psi^T\gamma^9 D_\theta\psi - \frac{1}{2\pi R^{3/2}}\,\frac{1}{2}\,\psi^T\gamma^I[x^I,\, \psi]\Bigg]\,,
\end{aligned} \tag{2.24}$$

[2] For finite N, the precise form of the mapping rule depends on whether N is even or odd, n is smaller, equal or larger than $N/2$ [8]. But any of its details are not necessary to taken into account here explicitly.

where the covariant derivatives and $f_{\tau\,\theta}$ are respectively

$$
\begin{aligned}
f_{\tau\,\theta} &\equiv \partial_\tau a_\theta - \partial_\theta a_\tau + i\frac{1}{2\pi}\left[a_\tau,\, a_\theta\right], \\
a_\tau &\equiv a_t\,, \\
D_a\phi &\equiv \partial_a\phi + i\frac{1}{2\pi}\left[a_a,\, \phi\right] \qquad (a=\tau,\, \theta)\,.
\end{aligned}
\tag{2.25}
$$

The action (2.24) coincides with the action of matrix string theory [9]. Hence, the quantum wrapped membrane will be defined by matrix string theory with the matrix variables obeying the twisted boundary condition (2.17), instead of the periodic one used in the original matrix string theory [9].

3. Sekino-Yoneya procedure from D-branes

In this section, we would like to rederive the Sekino-Yoneya procedure using the M(atrix) theory techniques. We start from matrix string theory. Matrix string theory is basically the theory of $D0$-branes with one space-like direction compactified. It was originally invented by following figure 1.

Since type-IIA strings can be considered as wrapped membranes, the matrix regularization by Sekino and Yoneya may be derived by closely examining this figure. We will show that this is indeed the case and the variables on the wrapped membrane are precisely expressed by the rules (2.18), (2.19) in terms of the matrix string variables.

Let us start with a collection of MN $D0$-branes. The variables on the world volume corresponding to such branes are $MN\times MN$ hermitian matrices. An $MN\times MN$ matrix F can be expressed as

$$
F=\sum_{\vec{m}} f_{\vec{m}}\, J^{(MN)}_{\vec{m}}\,,
\tag{3.1}
$$

where $\{J^{(MN)}_{\vec{m}}\}_{\vec{m}}$ is a basis of $MN\times MN$ matrices defined by eq. (A.3) in appendix, and $\vec{m}\equiv(m_1,m_2)$ with $-MN/2<m_1,\, m_2\leq MN/2$. Considering $J^{(MN)}_{\vec{m}}$ to be a regularized version of $e^{im_1\sigma^1+im_2\sigma^2}$, we can get the matrix regularization.

We would like to express a membrane wrapped around a compactified direction x^9 using these $D0$-branes. We will regard these MN $D0$-branes as M copies of N $D0$-branes,

$$
\begin{tikzcd}
\text{D0} \arrow[rr, leftrightarrow, "T_9"] \arrow[d, leftrightarrow, "9\text{--}11\text{ flip}"'] & & \text{D1} \arrow[d, leftrightarrow, "\text{S duality}"] \\
\text{IIA string} \arrow[rr, leftrightarrow, "T_{11}"'] & & \text{IIB string}
\end{tikzcd}
$$

Figure 1: Connection between type-IIA strings and $D0$-branes via duality transformations. T_μ denotes the T-dual transformation along the x^μ-direction.

consider these copies as the mirror images due to the torus compactification, and take the limit $M \to \infty$ eventually. Accordingly, we express an $MN \times MN$ matrix F in the form of an $M \times M$ matrix F_{ab}, $(a, b = 1, \ldots, M)$ with each matrix element F_{ab} being an $N \times N$ matrix. Thus the matrix variables $A_0(\tau)$, $x^I(\tau)$ $(I = 1, \cdots, 8)$, $x^9(\tau)$ (which becomes the extra dimension $x^{\#}$ after the $9 \leftrightarrow \#$ flip), $\psi(\tau)$ on the world volume satisfies the **Z**-orbifolding condition [14]

$$\begin{aligned} \phi_{ab}(\tau) &= \phi_{a-1,\,b-1}(\tau)\,, \\ x^9_{ab}(\tau) &= x^9_{a-1,\,b-1}(\tau) + 2\pi R \delta_{ab}\,, \end{aligned} \tag{3.2}$$

where we express the variables other than x^9 as ϕ collectively. The configuration of a membrane wrapped around the compactified direction x^9 can be expressed as $x^9(\tau) = MRQ_{(MN)}$, where $Q_{(MN)}$ is an $MN \times MN$ matrix defined by

$$Q_{(MN)} = \begin{pmatrix} 0 & & & \\ & \frac{2\pi}{MN} & & \\ & & \ddots & \\ & & & \frac{2\pi}{MN}(MN-1) \end{pmatrix}. \tag{3.3}$$

This obviously satisfies the boundary condition in eq. (3.2). Moreover, since $e^{iQ_{(MN)}} = V_{(MN)} \sim e^{i\sigma^2}$, this configuration can be considered as the matrix version of $x^9(\tau) = R\rho$ with $\rho = M\sigma^2$. Thus we take the variable $x^9(\tau)$ to be the sum of the background and the fluctuations around it:

$$\begin{aligned} x^9(\tau) &= MRQ_{(MN)} + \hat{x}^9(\tau)\,, \\ \hat{x}^9_{ab}(\tau) &= \hat{x}^9_{a-1,\,b-1}(\tau)\,. \end{aligned} \tag{3.4}$$

Now a matrix $F = \sum_{\vec{m}} f_{\vec{m}}\, J^{(MN)}_{\vec{m}}$ is interpreted as a function $f(\sigma, \rho)$ depending on two coordinates σ, ρ defined as

$$f(\sigma, \rho) = \sum_{\vec{m}} f_{\vec{m}}\, e^{im_1\sigma + im'_2\rho}\,, \tag{3.5}$$

with $m'_2 = m_2/M$ and $\sigma = \sigma^1$. As we will see, the condition (3.4) requires that m_2 is divisible by M. Therefore, in the expansion (3.5), ρ can be considered to be discretized in unit of $2\pi/N$. Under this rule, the commutators of matrix variables are mapped to the brackets $\{,\}$ with a normalization factor.

In order to get the matrix string variables, we need to T-dualize the above matrix variables. Following the familiar procedure [14], we introduce $N \times N$ matrices $\widetilde{\Phi}(\tau, \tilde{x})$, $\widetilde{A}_{\tilde{x}}(\tau, \tilde{x})$ as

$$\begin{aligned} \widetilde{\Phi}(\tau, \tilde{x}) &\equiv \sum_a e^{ia\tilde{x}/\widetilde{R}} \phi_{0a}(\tau)\,, \\ \widetilde{A}_{\tilde{x}}(\tau, \tilde{x}) &\equiv \sum_a e^{ia\tilde{x}/\widetilde{R}} \left(\frac{1}{2\pi\alpha'} x^9_{0a}(\tau) \right). \end{aligned} \tag{3.6}$$

Here $\tilde{x}$, $(\tilde{x} \approx \tilde{x} + 2\pi\widetilde{R})$ is the coordinate on the dual torus and $\widetilde{R} \equiv \alpha'/R$. In this representation, $x^9(\tau)$ is replaced by a differential operator $2\pi\alpha'(i\partial_{\tilde{x}} + \widetilde{A}_{\tilde{x}}(\tau, \tilde{x}))$. Usually $\widetilde{A}_{\tilde{x}}$, $\widetilde{\Phi}$ thus obtained yield the variables in matrix string theory. In our case, from eq. (3.4), we get $x^9_{00}(\tau) = RQ_{(N)} + \hat{x}^9_{00}(\tau)$, where $Q_{(N)}$ is given by eq. (3.3) with MN replaced by N. Thus $\widetilde{A}_{\tilde{x}}$ defined above contains a background piece proportional to $Q_{(N)}$. This background can be eliminated by a global gauge transformation, which changes the boundary conditions of the variables. Namely we define the new variables as follows:

$$\begin{aligned} A_{\tilde{x}} &= \exp\left(-i\frac{\tilde{x}}{2\pi\widetilde{R}}Q_{(N)}\right)\left(i\partial_{\tilde{x}} + \widetilde{A}_{\tilde{x}}\right)\exp\left(i\frac{\tilde{x}}{2\pi\widetilde{R}}Q_{(N)}\right), \\ \Phi &= \exp\left(-i\frac{\tilde{x}}{2\pi\widetilde{R}}Q_{(N)}\right)\widetilde{\Phi}\exp\left(i\frac{\tilde{x}}{2\pi\widetilde{R}}Q_{(N)}\right). \end{aligned} \tag{3.7}$$

In order to compare the notation used in the previous section, we define

$$\tilde{x} = \widetilde{R}\theta\,, \qquad A_{\tilde{x}}(\tilde{x}) = \frac{1}{\widetilde{R}}A_\theta(\theta)\,, \tag{3.8}$$

and consider A_θ, Φ to be functions of θ. A_θ, Φ obey the twisted boundary condition:

$$\begin{aligned} A_\theta(\theta + 2\pi) &= \left(V_{(N)}\right)^{-1} A_\theta(\theta)\, V_{(N)}\,, \\ \Phi(\theta + 2\pi) &= \left(V_{(N)}\right)^{-1} \Phi(\theta)\, V_{(N)}\,, \end{aligned} \tag{3.9}$$

where $V_{(N)} = \exp(iQ_{(N)})$. By a unitary transformation $A_\theta \mapsto (S_{(N)})^\dagger A_\theta S_{(N)}$ and $\Phi \mapsto (S_{(N)})^\dagger \Phi S_{(N)}$, with

$$S_{(N)} = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_{(N)} & \omega_{(N)}^2 & \cdots & \omega_{(N)}^{N-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & \omega_{(N)}^{N-1} & \omega_{(N)}^{2(N-1)} & \cdots & \omega_{(N)}^{(N-1)^2} \end{pmatrix}, \tag{3.10}$$

satisfying

$$\begin{cases} (S_{(N)})^\dagger V_{(N)} S_{(N)} = \left(U_{(N)}\right)^{-1} \\ (S_{(N)})^\dagger U_{(N)} S_{(N)} = V_{(N)} \end{cases}, \tag{3.11}$$

we can turn the boundary condition into

$$\begin{aligned} A_\theta(\theta + 2\pi) &= U_{(N)} A_\theta(\theta)\, \left(U_{(N)}\right)^{-1}, \\ \Phi(\theta + 2\pi) &= U_{(N)} \Phi(\theta)\, \left(U_{(N)}\right)^{-1}, \end{aligned} \tag{3.12}$$

which coincide with the one (2.17) satisfied by the matrix string variables appeared in the previous section.

Now what we should do is to show how the variables obtained via the above procedure is related to the world volume variables on the membranes, simply following what we have done. Let F_{rs}, $(r, s = 1, \ldots, MN)$ denotes the matrix element of an $MN \times MN$ matrix F. From eq. (3.2), we need matrices F satisfying the condition

$$F_{r+N,\, s+N} = F_{rs}\,, \tag{3.13}$$

to represent the configurations of $D0$-branes. Since the matrix elements of $J^{(MN)}_{\vec{m}}$ are

$$\left(J^{(MN)}_{\vec{m}}\right)_{rs} = (\omega_{MN})^{m_2(-\frac{m_1}{2}+s-1)}\,\delta^{(MN)}_{r-s,\,-m_1}\,, \tag{3.14}$$

where

$$\delta^{(MN)}_{rs} = \begin{cases} 1 & r \equiv s \bmod MN \\ 0 & \text{otherwise} \end{cases}\,, \tag{3.15}$$

the condition (3.13) for $F = J^{(MN)}_{\vec{m}}$ is satisfied if and only if m_2 is divisible by M, i.e., $m_2 = m'_2 M$ with $-N/2 < m'_2 \leq N/2$. Hence, F obeying the condition (3.13) is in general written by

$$F = \sum_{m_1,\, m'_2} f_{(m_1,\, m'_2 M)}\, J^{(MN)}_{(m_1,\, m'_2 M)}\,, \tag{3.16}$$

where the sum runs over $-N/2 < m'_2 \leq N/2$, and $-MN/2 < m_1 \leq MN/2$.

After the T-dualization (3.6), we should take $N \times N$ matrices $\{J^{(N)}_{\vec{m}}(\theta)\}_{\vec{m}}$ whose $(k,\, l)$-element $(k, l = 1, \cdots, N)$ is

$$\left(J^{(N)}_{\vec{m}}(\theta)\right)_{kl} \equiv \sum_a e^{ia\theta}\left(J^{(MN)}_{\vec{m}}\right)_{k,\,l+aN}\,, \tag{3.17}$$

as the basis for the matrix string variables satisfying the periodic boundary conditions. Accordingly the $MN \times MN$ matrix F in eq. (3.16) is mapped to

$$\begin{aligned} F &= \sum_{m_1,\, m'_2} f_{(m_1,\, m'_2 M)}\, J^{(MN)}_{(m_1,\, m'_2 M)} \\ \mapsto \tilde{f}(\theta) &= \sum_{m_1,\, m'_2} f_{(m_1,\, m'_2 M)}\, J^{(N)}_{(m_1,\, m'_2 M)}(\theta)\,, \end{aligned} \tag{3.18}$$

whose matrix elements are calculated as

$$\tilde{f}_{kl}(\theta) = \sum_{m_1,\, m'_2} f_{(m_1,\, m'_2 M)} \sum_a \delta_{k-l-aN,\,-m_1} \exp\left(\frac{2\pi i}{N} m'_2 \left(\frac{m_1}{2} + k - 1\right) + ia\theta\right). \tag{3.19}$$

Here the range of summation over a is chosen so that $-MN/2 < m_1 = l-k+aN \leq MN/2$.

To map $\tilde{f}(\theta)$ to the field $f(\theta)$ obeying the twisted boundary condition in eq. (3.9), we apply the transformation in eq. (3.7) to $\tilde{f}(\theta)$ and we get

$$f(\theta) = e^{-i\frac{\theta}{2\pi}Q_{(N)}}\,\tilde{f}(\theta)\,e^{i\frac{\theta}{2\pi}Q_{(N)}}\,. \tag{3.20}$$

Substituting eq. (3.19) into this, we can get the matrix element of $f(\theta)$. Since $-MN/2 < m_1 \le MN/2$ can be written as $m_1 = m_1' + a'N$ with $-N/2 < m_1' \le N/2$ and $-M/2 < a' \le M/2$, we can decompose the summation over m_1 into those of m_1' and a' and obtain

$$f_{kl}(\theta) = \sum_{m_1', m_2'} \left(\sum_{a'} f_{(m_1'+a'N, m_2'M)} e^{ia'(\theta+\pi m_2')} \right) \left(J_{\vec{m}'}^{(N)} \right)_{kl} e^{\frac{i}{N} m_1' \theta} . \tag{3.21}$$

Performing the unitary transformation $S_{(N)}$ in eq. (3.10), we get

$$\begin{aligned} f_{kl}(\theta) &= \sum_{m_1', m_2'} \sum_{a'} f_{(m_1'+a'N, m_2'M)} \delta^{(N)}_{k-l, m_2'} \exp\left(\frac{i}{N} (m_1' + a'N) \left(\pi(-m_2' + 2k - 2) + \theta \right) \right) \\ &= \sum_{m_1, m_2'} f_{(m_1, m_2'M)} \delta^{(N)}_{k-l, m_2'} \exp\left(\frac{i}{N} m_1 \left(\pi(-m_2' + 2k - 2) + \theta \right) \right) . \end{aligned} \tag{3.22}$$

Since the summand depends only on the combination $m_1 = m_1' + a'N$, the summation over m_1' and a' on the first line amounts to a summation over m_1. Now let us compare this last expression with the function (3.5) corresponding to the matrix variable in the limit $M \to \infty$. Then we can read off $f_{kl}(\theta)$ as

$$f_{kl}(\theta) = \frac{1}{N} \sum_{\rho} \sum_{m_2'} e^{-im_2'\rho} \, \delta^{(N)}_{k-l, m_2'} f \left(\sigma = \frac{-m_2' + 2k - 2}{N} \pi + \frac{\theta}{N} , \rho \right) . \tag{3.23}$$

In order to obtain Sekino-Yoneya mapping rule (2.19), we assume that N is even and restrict m_2 to be even. Then we get

$$f_{kl}(\theta) = \frac{1}{N} \sum_{\rho} e^{-i(k-l)\rho} f \left(\sigma = \frac{k + l - 2}{N} \pi + \frac{\theta}{N} , \rho \right) , \tag{3.24}$$

which coincides with the rule given by Sekino and Yoneya. We note that the modulo π periodicity along σ is also derived to perform the sum over m_2' consistent with modulo N periodicity. Restricting m_1 to be even does not matter as a regularization, but it may cause some trouble because we should discard some of the degrees of freedom on the world volume of $D0$-branes. One way to get the rule similar to Sekino-Yoneya mapping rule without discarding the odd modes is to use a different basis of matrices:

$$J'^{(N)}_{\vec{m}} \equiv \left(\omega_{(N)} \right)^{-m_1 m_2} \left(U_{(N)} \right)^{2m_1} \left(V_{(N)} \right)^{m_2} . \tag{3.25}$$

Assuming N to be odd, $J'^{(N)}_{\vec{m}}$ can be used as a basis of $N \times N$ matrices. Using $J'^{(N)}$ in the above procedure assuming N and M to be odd, we get a mapping rule

$$f_{kl}(\theta) = \frac{1}{N} \sum_{\rho} e^{-i(k-l)\rho} f \left(\sigma = 2 \left(\frac{k + l - 2}{N} \pi + \frac{\theta}{N} \right) , \rho \right) . \tag{3.26}$$

This relation gives a one-to-one correspondence at finite M, but after $M \to \infty$ to get matrix string theory, matrices becomes a double-cover of the membrane variables.

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4. Super-Poincaré symmetry in the short distance regime

4.1 remark on dynamics of regularized membrane

As we have established in the previous section, matrix string theory describes wrapped $M2$-branes in M theory in the DLCQ limit. Here let us notice that the time coordinate τ used in the matrix string action (2.25) is not precisely σ^0 used in the membrane action. From eqs. (2.12) and (2.23), $\tau \sim RN\sigma^0$. Thus, the natural time scale $\sigma^0 \sim \mathcal{O}(1)$ of the membrane corresponds to $\tau \sim \mathcal{O}(NR)$. Since we are interested in the limit $N \to \infty$, this fact implies that the very infrared dynamics of matrix string theory describes the world volume theory of wrapped membranes. Matrix string theory is basically a two dimensional gauge theory, and it becomes strongly coupled in the infrared. Hence, in order to describe wrapped membranes, we need to study the strongly coupled gauge theory. Let us see this more explicitly in the following.

As a regularization of the world volume theory of membranes, all the variables are replaced by matrices in matrix string theory. Products of matrices become products of corresponding variables in the limit $N \to \infty$ and commutators are proportional to brackets $\{,\}$. For a finite N, the terms appearing in the matrix string action can be rewritten in terms of the variables appearing in the membrane action using the following formula [8]:

$$\begin{aligned}
&\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{N} \operatorname{tr}\left(\phi_{(1)}(\theta)\phi_{(2)}(\theta)\cdots\phi_{(p)}(\theta)\right) = \\
&\quad = \frac{1}{N}\sum_{\rho}\int_0^{2\pi}\frac{d\sigma}{2\pi}\exp\left(-i\frac{\pi}{N}\sum_{1\le j<k\le p}\left(\partial_{\sigma_{(j)}}\partial_{\rho_{(k)}} - \partial_{\rho_{(j)}}\partial_{\sigma_{(k)}}\right)\right)\times \\
&\qquad \times \Phi_{(1)}(\sigma_{(1)},\,\rho_{(1)})\Phi_{(2)}(\sigma_{(2)},\,\rho_{(2)})\cdots\Phi_{(p)}(\sigma_{(p)},\,\rho_{(p)})\big|_{\sigma_{(j)}=\sigma,\,\rho_{(j)}=\rho}\,, \\
&\quad \equiv \frac{1}{N}\sum_{\rho}\int_0^{2\pi}\frac{d\sigma}{2\pi}\left(\Phi_{(1)}\star\Phi_{(2)}\star\cdots\star\Phi_{(p)}\right)(\sigma,\,\rho)\,,
\end{aligned} \tag{4.1}$$

where ∂_ρ is a difference operator. Using the noncommutative $\star$-product defined above, the action (2.21) can be written in terms of the variables on the world volume of membranes as

$$\begin{aligned}
I_{\mathrm{WM}} = \int dt \int_0^{2\pi}\frac{d\sigma}{2\pi}\frac{1}{N}\sum_{\rho}\Big[&\frac{R^3}{2}(F_{t\,\sigma})^2 + \frac{1}{2}(D_t X^I)^2 - \frac{1}{2}(D_\sigma X^I)^2 - \\
&- \frac{N^2}{(2\pi)^2R^3}\frac{1}{4}\left(i[X^I,\,X^J]\right)^2 - \\
&- \frac{i}{2}\,\Psi^T D_t\Psi - \frac{i}{2}\,\Psi^T\gamma^9 D_\sigma\Psi - \frac{N}{2\pi R^{3/2}}\frac{1}{2}\Psi^T\gamma^I[X^I,\,\Psi]\Big]_\star\,.
\end{aligned}$$

$$\begin{aligned}
F_{t\,\sigma} &\equiv \partial_t A_\sigma - \partial_\sigma A_t + i\frac{N}{2\pi}[A_t,\,A_\sigma]_\star\,,\\
D_a\Phi &\equiv \partial_a\Phi + i\frac{N}{2\pi}[A_a,\,\Phi]_\star \qquad (a = t,\,\sigma)\,.
\end{aligned} \tag{4.2}$$

After a rescaling $A_a \to A_a/R^{3/2}$ to make the kinetic term for the gauge field canonical, the coupling constant of the theory regularizing the wrapped membrane is naively identified as

$$\frac{N}{2\pi\, R^{3/2}}\,. \tag{4.3}$$

However, one should notice a subtlety coming from the fact that the kinetic terms in this action lack derivatives in the ρ-direction. This implies that the free propagator of X^μ (and A_t, A_σ) depends on the discretized ρ-coordinate through the Kronecker-delta δ_{mn}. In the perturbative expansion using such propagators, one can easily see that the situation is similar to the large-N theory and the coupling constant becomes effectively the t'Hooft coupling

$$g_{\mathrm{WM}} = \frac{1}{2\pi}\left(\frac{N}{R}\right)^{3/2}\,. \tag{4.4}$$

On the other hand, the t'Hooft coupling constant in the matrix string action (2.24) is identified as

$$\begin{aligned} g_{\mathrm{MS}} &= \frac{N^{1/2}}{2\pi R^{3/2}} \\ &= \frac{g_{\mathrm{WM}}}{N}\,. \end{aligned} \tag{4.5}$$

The action in eq. (4.2) was obtained by rewriting the regularized action (2.24) using the $\star$-product. The coupling constants of these theories have the dimension of the inverse of length and the relation between coupling constants reflects the fact that the coordinates τ, θ used in the matrix string action is obtained by dividing σ^j used in the membrane action by N essentially. For the world sheet length scale l of our interest, the dimensionless effective coupling constant g_{eff} is

$$\begin{aligned} g_{\mathrm{eff}} &= l g_{\mathrm{MS}} \\ &= \frac{l}{2\pi}\frac{N^{1/2}}{R^{3/2}}\,. \end{aligned} \tag{4.6}$$

Thus, as long as we are concerned with the length scale,

$$l \leq \mathcal{O}\left(2\pi \frac{R^{3/2}}{N^{1/2}}\right), \tag{4.7}$$

we can carry out perturbative expansion in g_{eff} and observe the ultraviolet aspects of matrix string theory.

In the rest of this paper, we would like to examine if the super-Poincaré algebra closes for matrix string theory by studying this theory perturbatively. Actually we only consider the lowest order in the perturbation theory. Since this theory is supposed to describe M theory, we should require this as a consistency condition of the theory. We cannot get any information on the super-Poincaré invariance of the gauge fixed world volume theory of the wrapped membranes from such analysis (See ref. [18, 19, 21] for the existence of critical dimension of the membrane and the related topics.), because we should look at the strongly coupled region of the theory to do so. However, the correspondence between the membrane action and the matrix string action is useful when we try to find the super-Poincaré generators for matrix string theory, because, while matrix string theory itself does not give any intrinsic definition of such generators involving light-cone directions, the wrapped membrane action is classically super-Poincaré invariant.

4.2 Matrix string in modified light-cone gauge

Now let us study the matrix string action (2.24) at the lowest order in the perturbation theory. Namely we put $g_{\rm MS} = 0$. In the wrapped membrane action (4.2) equivalent to this action, putting $g_{\rm MS} = 0$ corresponds to putting $g_{\rm WM} = 0$. Then there are no terms involving derivatives in the ρ direction and we get N independent strings lying on the world volume of the membrane. Thus, naively, we can expect that the super-Poincaré generators are given as a sum of those for these strings. However actually these strings are not completely independent in the light-cone gauge fixing described in section 2. Indeed P^+ is taken to be constant which means that each string carries the same amount of p^+. Therefore the light-cone zero modes p^+, x^- should be the same for each string, which makes it difficult to construct some of the super-Poincaré generators. From the point of view of matrix string theory, this is consistent because there is only one p^+ and no notion of p^+ for each string. However if one looks at the theory more closely, one can see that there exists degrees of freedom from which we can construct p^+ and x^- for each string.

One may also worry about the pair x^+, p^-. In the light-cone gauge, we should take the gauge so that x^+ is common to all the strings. However this does not cause any problems in constructing the super-Poincaré generators, contrary to the p^+, x^- case.

In this subsection, we would like to consider the *modified light-cone gauge*, in which strings have independent p^+, x^-. We will show that the variables in such a gauge can be written in terms of those in the light-cone gauge. Thus the super-Poincaré generators in this new gauge can be expressed by the light-cone variables and we can calculate the commutators of these operators by using such expressions.

The action (2.24) possesses the gauge symmetry

$$\delta_\epsilon A_a = D_a \epsilon\,, \tag{4.8}$$

which can be considered as a regularized version of the area preserving diffeomorphism. The most convenient gauge choice would be $A_\sigma = 0$, but since the σ-direction is compact, we cannot fix the modes in A_σ which are constant with respect to σ. Therefore the best we could do is to take $\partial_\sigma A_\sigma = 0$ so that we have degrees of freedom corresponding to $A_\sigma(\rho)$. Actually we can reconstruct the variables corresponding to p^+ for each string from these degrees of freedom.

Similarly to matrix string theory, after taking the gauge $P^+ = 0$, we cannot fix $A_1 = 0$ using the area preserving diffeomorphism and the best we can do is to take $\partial_1 A_1 = 0$. In order to see which quantities should be identified as the p^+ for the strings, let us replace the gauge condition $P^+ = 0$ by $\partial_1 P^+ = 0$. This implies that $P^+ = P^+(\sigma^2)$ remains as a canonical variable and the strings put at the sites σ^2 have independent longitudinal momenta $P^+(\sigma^2)$. In this gauge, contrary to the light-cone gauge, we can take

$$A_1(\vec{\sigma}) = 0\,, \tag{4.9}$$

using the residual diffeomorphism.

Thus, at least in the continuum membrane theory, we can take the modified light-cone gauge in which the strings put at the sites along the ρ direction have independent p^+'s. Since this is just another gauge fixing, we can express the variables in such a gauge in terms of the variables in the light-cone gauge. Once such a relation is established, we will be able to find the super-Poincaré generators in the light-cone gauge from the ones constructed as a sum of those for independent strings in the modified light-cone gauge.

Classically it is straightforward to rewrite the variables in the modified light-cone gauge in terms of the variables in the light-cone gauge. In order to distinguish the variables in two gauges, let $\phi_{\mathrm{ml}}(\sigma^1, \sigma^2)$ denote the variables in the modified light-cone gauge and $\phi_{\mathrm{lc}}(\sigma, \rho)$ denote those in the light-cone gauge. Here $\sigma^1 = \sigma$ while σ^2 and ρ are related through the equation

$$\sigma^2 = \rho - A_{1\,\mathrm{lc}}(\rho)\,. \tag{4.10}$$

$p^+_{\mathrm{ml}}(\sigma^2)$ can be obtained as

$$p^+_{\mathrm{ml}}(\sigma^2) = \frac{P^+_{\mathrm{lc}}}{1 - \partial_\rho A_{1\,\mathrm{lc}}(\rho)}\,. \tag{4.11}$$

$x^-_{\mathrm{ml}}(\sigma^2)$ which is the canonical conjugate of $p^+_{\mathrm{ml}}(\sigma^2)$ can be given as

$$x^-_{\mathrm{ml}}(\sigma^2) = \int_0^{2\pi} \frac{d\sigma}{2\pi} X^-_{\mathrm{lc}}(\sigma, \rho(\sigma^2))\,, \tag{4.12}$$

where $X^-_{\mathrm{lc}}(\sigma, \rho)$ is given by solving the momentum constraints in the light-cone gauge. Other variables can be given as

$$\begin{aligned} X^I_{\mathrm{ml}}(\sigma^1, \sigma^2) &= X^I_{\mathrm{lc}}(\sigma^1, \rho)\,, \\ \theta_{\mathrm{ml}}(\sigma^1, \sigma^2) &= \theta_{\mathrm{lc}}(\sigma^1, \rho)\,, \\ P^I_{\mathrm{ml}}(\sigma^1, \sigma^2) &= P^I_{\mathrm{lc}}(\sigma^1, \rho)\frac{1}{1 - \partial_\rho A_{1\,\mathrm{lc}}(\rho)}\,, \end{aligned} \tag{4.13}$$

Thus classically all the variables can be rewritten in terms of those in the light-cone gauge. Since we would like to calculate the commutation relations of super-Poincaré generators using the canonical commutation relations of those variables, we should check if these relations can be made into quantum mechanical relations, in such a way that the canonical commutation relations are satisfied. What we are doing is basically a coordinate transformation $\rho \mapsto \sigma^2 = \rho - A_{1\,\mathrm{lc}}(\rho)$. Thus for quantities which do not involve the momentum P_{A_1} conjugate to $A_{1\,\mathrm{lc}}(\rho)$, there is no ordering ambiguity, we can regard $A_{1\,\mathrm{lc}}(\rho)$

as a c-number. The only variable about which we should be a little bit careful is $x^-_{\rm ml}(\sigma^2)$. Since $X^-_{\rm lc}$ on the right hand side of eq. (4.12) involves P_{A_1} we should fix the ordering of the operators so that the canonical commutation relations are satisfied in the modified light-cone gauge. Doing so is not difficult because P_{A_1} appears only linearly in $X^-_{\rm lc}$. By specifying the ordering so that we put P_{A_1} on the right of all the other operators, one can show that the commutation relations between operators involving P_{A_1} at most linearly are the same as their classical counterparts. Therefore what we should do is to check if the Poisson bracket

$$[x^-_{\rm ml}(\sigma^2),\, p^+_{\rm ml}(\sigma'^2)]_P = \frac{1}{2\pi}\delta(\sigma^2 - \sigma'^2)\,, \qquad (4.14)$$

and others are satisfied classically or not. Showing eq. (4.14) is straightforward but needs some care. Since σ^2 and ρ are related to each other as eq. (4.10), ρ can have nontrivial commutators with other variables, if we consider σ^2 as a c-number. Anyway, from the commutator of the momentum constraints and other variables, one can show that the variables in the modified light-cone gauge satisfy the canonical commutation relations.

Thus we obtain the relations between the variables of the continuum membrane theory in two different gauges. What we actually need is a discretized version of such relations. Namely, we would like to obtain world sheet variables with independent p^+, x^- in terms of the matrix string variables. This can be done as follows. Let us recall that the relations (4.13) give a canonical transformation classically, and a unitary transformation in the quantum theory. Therefore, we can get a unitary operator $\exp(i\mathcal{K})$ which translates the quantities at (σ, ρ) in the light-cone gauge into the quantities at (σ, σ^2) in the modified light-cone gauge. Thus if we discretize the operator $\mathcal{K}$ in an appropriate way, by applying the unitary transformation corresponding to $\exp(i\mathcal{K})$ to the matrix string variables, we can get the variables corresponding to N strings with independent p^+, x^-, which become those of modified light-cone gauge in the limit $N \to \infty$. Thus, in principle, we can construct the world sheet variables of N strings with independent p^+, x^- from those of matrix string theory.

Wrapped membranes with nontrivial spatial topologies yield global constraints associated with closed but non-exact generators of area preserving diffeomorphism [7, 20, 22]. Since the DLCQ limit needs tentative compactification of the light-cone direction x^-, a global constraint shows up in solving X^-. In this paper, our goal is a modest one, which is to realize membranes without any winding around x^-. Of course, in order to construct the full-fledged membrane theory, we need to consider the sectors with nontrivial winding. Since our main focus in this paper is the anomaly, it is sufficient to examine only the non-winding sector.

4.3 super-Poincaré symmetry of matrix string theory

With the explicit connection between two gauge choices, we can now see if there is super-Poincaré invariance at the leading order in matrix string theory by examining it in the modified light-cone gauge fixed membrane theory. To make contact with the ultraviolet limit discussed in Subsection 4.1, we perform the rescaling which leads to the action (2.11), together with the rescaled conjugate momenta Π^I given by

$$P^I = \frac{1}{R^{1/2}} \Pi^I . \tag{4.15}$$

The corresponding hamiltonian density $\mathcal{H}$ is obtained by rescaling $(-P^-)$ determined from $\phi_0 = 0$ as

$$-P^- = R\mathcal{H} . \tag{4.16}$$

The explicit form of $\mathcal{H}$ reads

$$\begin{aligned} \mathcal{H} = \frac{1}{P^+}\bigg(&\frac{1}{2}(\Pi^I)^2 + \frac{1}{2R^3}(P_{A_1})^2 + \frac{1}{2}(\partial_1 X^I)^2 + \frac{1}{4R^3}(\{X^I, X^J\})^2 + \\ &+ \frac{i}{2}\Psi^T\gamma^9\partial_1\Psi - i\frac{1}{R^{3/2}}\frac{1}{2}\Psi^T\gamma^I\{X^I, \Psi\}\bigg) . \end{aligned} \tag{4.17}$$

Let us define $\phi_{(m)}(\sigma^1)$ to be the value of a variable $\Phi(\sigma^1, \sigma^2 = ma)$ at the lattice site $\sigma^2 = ma$ ($a \equiv 2\pi/N$). As anticipated, the hamiltonian for $g_{\rm WM} = 0$ turns out to be just the sum of the hamiltonians of Green-Schwarz strings in the light-cone gauge;

$$\begin{aligned} H &= a\sum_m \int_0^{2\pi} d\sigma^1\, \mathcal{H} \\ &= a\sum_m \int_0^{2\pi} d\sigma^1\, \frac{1}{p^+_{(m)}}\bigg(\frac{1}{2}(\pi^I_{(m)}(\sigma^1))^2 + \frac{1}{2}(\partial_1 x^I_{(m)}(\sigma^1))^2 + \frac{i}{2}s^a_{(m)}(\sigma^1)\partial_1 s^a_{(m)}(\sigma^1) - \\ &\qquad - \frac{i}{2}s^{\dot a}_{(m)}(\sigma^1)\partial_1 s^{\dot a}_{(m)}(\sigma^1)\bigg) , \end{aligned} \tag{4.18}$$

where $\Psi(\vec\sigma)$ have been divided into two 8 dimensional spinors, $S^a(\vec\sigma)$, $S^{\dot a}(\vec\sigma)$.

At least at the zeroth order in $1/R^{3/2}$, the super-Poincaré generators would be basically obtained as the sum of the super-Poincaré generators of strings put along the σ^2-direction. The only thing one should take care is that since x^+ is common to all the strings, we take $p^-_{(m)}$ appearing in the generators to be the one obtained by solving the hamiltonian constraint, except for the Fourier zero mode with respect to m. For the zero mode, we use p^- conjugate to x^+ as in the usual string theory. With such a form, we can compute, say, $[L^{-I}, L^{-J}]$ in the similar way as in the string case [23] and find

$$\begin{aligned} [L^{I-}, L^{J-}] = a\sum_m \frac{1}{(p^+_{(m)})^2}\sum_{r=1}^\infty \bigg(&\alpha^I_{(m)\,-r}\alpha^J_{(m)\,r} - \alpha^J_{(m)\,-r}\alpha^I_{(m)\,r} + \tilde\alpha^I_{(m)\,-r}\tilde\alpha^J_{(m)\,r} - \\ &- \tilde\alpha^J_{(m)\,-r}\tilde\alpha^I_{(m)\,r}\bigg) \times \frac{1}{a} \times 2\left(\frac{\delta^{KK}}{8} - 1\right) r . \end{aligned} \tag{4.19}$$

Therefore the Schwinger term vanishes for $\delta^{KK} = 8$.

The calculation can be repeated for the supermembranes in the space-time with dimension equal to $D = 4$, 5 or 7, taking Majorana and/or Weyl properties into account. In those cases, as expected, Lorentz anomaly exists. Therefore the critical dimension for matrix string theory at the leading order is $D - 1 = 10$.

5. Conclusions and discussions

In this paper, we have shown that wrapped membranes can be naturally related to matrix string theory via the rule proposed by Sekino and Yoneya, using the M(atrix) theory techniques. From such a point of view, the membrane can be naturally regarded as a collection of strings interacting with each other. In the lowest order in the perturbative expansion of matrix string theory, these strings become free. With a change of variables discussed in section 4, the super-Poincaré generators of the theory can be constructed essentially as a sum of those of free strings and we can show that the theory is invariant under the super-Poincaré symmetry (except for the wrapped direction, x^9) if $D = 10$ for matrix string theory or $D = 11$ for the wrapped membrane.

Of course, this result is not sufficient to conclude that membrane theory is consistent only when $D = 11$. Since the matrix string action is that of a two dimensional gauge theory, the coupling constant is a dimensionful one. Therefore in order to see if the theory is invariant under the super-Poincaré symmetry, we should check this for all orders in the perturbation theory. It is quite challenging to see if we can construct the Lorentz generators at the next order of $1/R^{3/2}$.

We can also apply our approach to other types of membranes, e.g., open membranes with the various types of boundary conditions. The open membranes related to the heterotic strings in the context of M theory [24] will be regularized by $O(N)$ matrices (See ref. [25, 26, 27]). In this context, the membrane may be regarded as being composed of the interacting strings put at the sites along the spatial direction with boundary. It is interesting to see if the consistency conditions of the theory require the existence of appropriate number of fermions on each boundary which should provide E_8 gauge bosons.

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A. A basis of generators of $\mathrm{U}(N)$

This appendix summarizes the convention for t'Hooft matrices and the related objects used in the text. We assume that the rank N of the matrices treated is even.[3] Using $\omega_{(N)} \equiv \exp\left(i\frac{2\pi}{N}\right)$, $N \times N$ t'Hooft matrices $U_{(N)}$, $V_{(N)}$ satisfy the relation

$$U_{(N)}V_{(N)} = \omega_{(N)}V_{(N)}U_{(N)} \,. \tag{A.1}$$

[3] See, for instance, ref. [28].

One explicit representation of such $U_{(N)}$, $V_{(N)}$ is

$$U_{(N)} = e^{iP_{(N)}} = \begin{pmatrix} 0\,1\,0 \cdots 0\,0\,0 \\ 0\,0\,1 \cdots 0\,0\,0 \\ 0\,0\,0 \cdots 0\,0\,0 \\ \cdot\,\cdot \\ \quad\cdot\,\cdot \\ \qquad\cdot\,\cdot \\ \qquad\quad\cdot\,\cdot \\ \qquad\qquad\cdot\,1 \\ 1\,0\,0 \cdots 0\,0\,0 \end{pmatrix},$$

$$V_{(N)} = e^{iQ_{(N)}} = \begin{pmatrix} 1 & & & & \\ & \omega_{(N)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \left(\omega_{(N)}\right)^{N-1} \end{pmatrix}. \tag{A.2}$$

They satisfy $(U_{(N)})^N = 1 = (V_{(N)})^N$.

A basis of Lie algebra of U(N) is given by $\left\{ J^{(N)}_{\vec{m}} \right\}_{\vec{m}}$, where $\vec{m} \equiv (m_1, m_2)$ is a set of two integers m_1, m_2 and

$$J^{(N)}_{\vec{m}} \equiv \left(\omega_{(N)}\right)^{-\frac{1}{2}m_1 m_2} \left(U_{(N)}\right)^{m_1} \left(V_{(N)}\right)^{m_2}. \tag{A.3}$$

One can check that $(J^{(N)}_{\vec{m}})^\dagger = J^{(N)}_{-\vec{m}}$, and

$$\begin{aligned} J^{(N)}_{\vec{m}} J^{(N)}_{\vec{n}} &= \left(\omega_{(N)}\right)^{\frac{1}{2}\vec{m}\times\vec{n}} J^{(N)}_{\vec{m}+\vec{n}}, \\ \left[J^{(N)}_{\vec{m}}, J^{(N)}_{\vec{n}} \right] &= 2i \sin\left(\frac{\pi}{N}\vec{m}\times\vec{n}\right) J^{(N)}_{\vec{m}+\vec{n}}, \end{aligned} \tag{A.4}$$

with $\vec{m}\times\vec{n} \equiv m_1 n_2 - m_2 n_1$.

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